ASYMPTOTIC REPRESENTATION OF THE SOLUTION OF THE FIRST BOUNDARY-VALUE PROBLEM FOR THE HEAT-CONDUCTION EQUATION WITH A MOVING BOUNDARY

A. P. Doroshenko

UDC 517.9

We obtain an asymptotic solution of the first boundary-value problem for the heat-conduction equation in a region with a moving boundary, which initially may degenerate to a point.

1. Statement of the Problem

We consider the problem of finding in the region $G[0 < t < T,\ 0 < x < \alpha(t)]$ a solution of the heat-conduction equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

satisfying the boundary conditions

$$u(+0, t) = \varphi(t), \quad u(\alpha(t) - 0, t) = \psi(t). \tag{2}$$

We assume that the function $\alpha(t)$, which describes the motion of the boundary, is positive, monotonically increasing, differentiable for $t > 0$, and vanishes at the point $t = 0$ only. We assume the functions $\varphi(t)$ and $\psi(t)$ are continuous.

We seek a solution of the problem (1), (2) in the form of a sum of thermal potentials of a double layer (see [1]),

$$u(x, t) = W_0[\nu(\tau)] + W[\mu(\tau)], \tag{3}$$

where

$$W_0[\nu(\tau)] = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4a^2(t-\tau)}\right] \nu(\tau)\, d\tau, \tag{4}$$

$$W[\mu(\tau)] = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{x-\alpha(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{(x-\alpha(\tau))^2}{4a^2(t-\tau)}\right] \mu(\tau)\, d\tau. \tag{5}$$

In order that the solution (3)-(5) satisfy the boundary conditions (2) the unknown densities $\nu(t)$ and $\mu(t)$ must be solutions of the system of integral equations considered in [2, 3]. For solvability of this system of integral equations we assume that $\nu(t) = O(k_1 t^{\beta_1})$, $\mu(t) = O(kt^{\beta})$, where β_1, $\beta > -1$; k_1, and k are nonzero constants.

2. Asymptotic Behavior of the Thermal Potentials

We consider a function $F(t)$ to belong to the class $M_\beta(k)$ if

$$\lim_{t\to+0} \frac{F(t)}{t^\beta} = k, \quad k = \text{const} \neq 0;$$

V. I. Lenin Khar'kov Polytechnic Institute, Khar'kov. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 2, pp. 352-357, August, 1975. Original article submitted March 3, 1974.

also, we have $F(t) \in M_\infty$, if $\lim\limits_{t\to+0} \frac{F(t)}{t^\beta} = 0$ for arbitrary β.

THEOREM. If $\mu(t) \in M_\beta(k)$, $\nu(t) \in M_{\beta_1}(k_1)$, where β, $\beta_1 > -1$, and $\alpha(t) \in M_\gamma(c)$, $\gamma > 1/2$, then: 1) $W[\mu(\tau)] \in M_\beta(-k)$, 2) $W_0[\nu(\tau)] \in M_{\beta_1}(k_1)$.

To prove the first part of the theorem we represent the function $\mu(t)$ in the form

$$\mu(t) = kt^\beta + \mu_1(t), \quad \text{where } \mu_1(t) = o(t^\beta).$$

Then

$$W[\mu(\tau)] = W[k\tau^\beta] + W[\mu_1(\tau)]. \tag{6}$$

Let us estimate the first term on the right side of Eq. (6). To simplify the discussion we take $\alpha(t) = ct^\gamma$. We transform the expression $W[k\tau^\beta]$ to the form

$$W[k\tau^\beta] = \frac{2kt^\beta}{\sqrt{\pi}} I(\omega), \tag{7}$$

where

$$I(\omega) = \int\limits_\omega^\infty \left[\delta - c\left(1 - \frac{\omega^2}{z^2}\right)^\nu\right]\left(1 - \frac{\omega^2}{z^2}\right)^\beta \exp\left\{-z^2 \times \left[\delta - c\left(1 - \frac{\omega^2}{z^2}\right)^\nu\right]^2\right\} dz,$$

$$\delta = \frac{x}{t^\nu}, \text{ where } 0 < \delta < c, \ \omega = \frac{t^{\nu-1/2}}{2a}.$$

Since $\gamma > 1/2$, it follows that $\omega \to 0$ as $t \to 0$. We now show that

$$\lim_{\omega\to0} I(\omega) = -\frac{\sqrt{\pi}}{2}. \tag{8}$$

To do this, we write $I(\omega)$ as a sum: $I(\omega) = I_1(\omega) + I_2(\omega)$. Each of the functions $I_1(\omega)$ and $I_2(\omega)$ may be expressed in terms of an integral similar to the integral for determining $I(\omega)$. For the integral $I_1(\omega)$ the region of integration is the interval $[\omega, s\omega]$; for the integral $I_2(\omega)$ it is the interval $(s\omega, +\infty)$, where s is a fixed quantity with $s > 1$.

The following estimate is valid:

$$|I_1(\omega)| < \frac{\omega s^3 c}{2} \int\limits_{\frac{1}{s^2}}^1 (1-\lambda)^\beta d\lambda = \frac{\omega s^3 c}{2(\beta+1)}\left(1 - \frac{1}{s^2}\right)^{\beta+1},$$

whence it follows that

$$\lim_{\omega\to0} I_1(\omega) = 0. \tag{9}$$

We choose the parameter s so that when $z \in [s\omega, \infty)$ the expression $\left[\delta - c\left(1 - \frac{\omega^2}{z^2}\right)^\nu\right] < 0$, i.e., the point z at which $\left[\delta - c\left(1 - \frac{\omega^2}{z^2}\right)^\nu\right]$ vanishes is located inside the interval $[\omega, s\omega]$. For each interior point $x[0 < x < \alpha(t)]$ there exist fixed quantities s and b, such that $\delta = x/t^\gamma \le c\,[1 - (1/s^2)]^\gamma - b$, where $s > 1$, $b > 0$. With s chosen in this way the second term $I_2(\omega)$ has the estimate

$$|I_2(\omega)| < (c-\delta)\int\limits_0^\infty \exp(-z^2b^2)\,dz = \frac{(c-\delta)\sqrt{\pi}}{2b} \quad \text{for } \beta \geqslant 0,$$

$$|I_2(\omega)| < (c-\delta)\left(1 - \frac{1}{s^2}\right)^\beta \int\limits_{s\omega}^\infty \exp(-z^2b^2)\,dz = \frac{(c-\delta)\sqrt{\pi}}{2b}\left(1 - \frac{1}{s^2}\right)^\beta \quad \text{for } -1 < \beta < 0.$$

Thus, the integral $I_2(\omega)$ converges uniformly with respect to the parameter ω. From this it follows that

$$\lim_{\omega\to 0} I_2(\omega) = -\frac{\sqrt{\pi}}{2}. \tag{10}$$

From the relations (9) and (10) we obtain the result (8). From the formulas (7) and (8) it follows that $W[k\tau^{\beta}]$ has the asymptotic property

$$W[k\tau^{\beta}] \in M_{\beta}(-k) \quad \text{for} \quad \beta > -1.$$

Since $\mu_1(t) = 0(t^{\beta})$ and $W[k\tau^{\beta}] \in M_{\beta}(-k)$, it follows that $W[\mu_1(\tau)] = 0(t^{\beta})$; therefore, the order of the thermal potential $W[\mu(\tau)]$ is determined by the order of the integral $W[k\tau^{\beta}]$, i.e., $W[\mu(\tau)] \in M_{\beta}(-k)$ for $\beta > -1$, which is what we wished to prove.

The second part of the theorem is proved in a similar way. The theorem holds even in the case when $\alpha(t)$ is an arbitrary function belonging to the class $M_{\gamma}(c)$, i.e., $\alpha(t) = ct^{\gamma} + \alpha_1(t)$, where $\alpha_1(t) = 0(ct^{\gamma})$.

3. Asymptotic Expansion of the Thermal Potential $W_0[k\tau^{\beta}]$

An asymptotic expansion of the function f(t) is given by the equation

$$f(t) = \sum_{n=0}^{\infty} h_n t^n + \varphi(t), \tag{11}$$

where

$$h_n = \frac{1}{n!} \lim_{t\to+0} \frac{d^n f}{dt^n}, \quad \lim_{t\to+0} \frac{\varphi^{(m)}(t)}{t^{\lambda}} = 0 \tag{12}$$

for arbitrary λ and m = 0, 1, 2, ... We consider

$$W_0[k\tau^{\beta}] = \frac{k}{2a\sqrt{\pi}} \int_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4a^2(t-\tau)}\right] \tau^{\beta}\, d\tau, \text{ where } \beta > -1,$$

and we find its asymptotic expansion as $t \to +0$. Another integral representation of the thermal potential is

$$W_0[k\tau^{\beta}] = \frac{2kt^{\beta}}{\sqrt{\pi}} I_0(\omega, \delta), \tag{13}$$

where

$$I_0(\omega, \delta) = \frac{1}{4} \int_0^1 \frac{y^{\beta}}{1-y} H_1\left(\frac{\omega\delta}{\sqrt{1-y}}\right) \exp\left(-\frac{\omega^2\delta^2}{1-y}\right) dy,$$

$$\delta = \frac{x}{t^{\gamma}}, \quad \omega = \frac{t^{\gamma-1/2}}{2a}, \quad H_1(x) = 2x.$$

Since $\gamma > 1/2$, then $\omega \to 0$ as $t \to +0$. We first construct an asymptotic power series for the function $I_0(\omega, \delta)$ as $\omega \to 0$. To do this, it is necessary, according to the relations (11) and (12), to find

$$\lim_{\omega\to 0} \frac{\partial^n I_0(\omega, \delta)}{\partial \omega^n} \qquad (n = 0, 1, 2, \ldots). \tag{14}$$

We can show that

$$\lim_{\omega\to 0} I_0(\omega, \delta) = \frac{\sqrt{\pi}}{2}, \tag{15}$$

and that the n-th derivative is given by the equation

$$\frac{\partial^n I_0(\omega,\ \delta)}{\partial\omega^n} = (-1)^n\ \frac{\delta^n}{2\omega^n} \int\limits_{\omega}^{\infty} z^{n-1}\left(1-\frac{\omega^2}{z^2}\right)^{\beta} H_{n+1}(\delta z)\ \exp\ (-\delta^2 z^2)\ dz, \tag{16}$$

where $H_n(x)$ is a Hermite polynomial. To determine the limit (14) we break the integral on the right side of Eq. (16) into two integrals (additivity property), taking the first of them over the interval $[\omega,\ s\omega]$ and the second over the interval $(s\omega,\ \infty)$, and we write

$$\frac{\partial^n I_0(\omega,\ \delta)}{\partial\omega^n} = I_n^{(1)}(\omega,\ \delta) + I_n^{(2)}(\omega,\ \delta). \tag{17}$$

The auxiliary parameter s, introduced above, must be larger than one. In $I_n^{(1)}(\omega,\ \delta)$ we make the substitution $\omega/z = y$ and then let $\omega \to 0$. We obtain

$$\chi_n = \lim_{\omega\to 0} I_n^{(1)}(\omega,\ \delta) = (-1)^n\ \frac{H_{n+1}(0)\,\delta^n}{2} \int\limits_{1/s}^{1} \frac{(1-y^2)^{\beta}}{y^{n+1}}\,dy.$$

Since $H_{2m+1}(0) = 0$ and $H_{2m}(0) = (-1)^m(2m)!/m!$, it follows that

$$\chi_n = \begin{cases} 0 \quad \text{for} \quad n = 2m, \\ (-1)^{m+1}\,2^{m-1}\,(2m-1)!!\,\delta^{2m-1} \int\limits_{1/s}^{1} (1-y^2)^{\beta}/y^{2m}dy \quad \text{for} \quad n = 2m-1. \end{cases} \tag{18}$$

We now find the limit of the second term $I_n^{(2)}(\omega,\ \delta)$ as $\omega \to 0$. Obviously,

$$\lim_{\omega\to 0}\left[\delta^n \int\limits_{s\omega}^{\infty} z^{n-1} H_{n+1}(\delta z)\ \exp\ (-z^2\delta^2)\left(1-\frac{\omega^2}{z^2}\right)^{\beta} dz\right] = \frac{1}{2}\int\limits_{-\infty}^{+\infty} v^{n-1} H_{n+1}(v)\ \exp\ (-v^2)\,dv = 0, \quad n \geqslant 1.$$

Applying l'Hospital's rule, we have for n = 2m

$$\sigma_{2m} = \lim_{\omega\to 0} I_{2m}^{(2)}(\omega,\ \delta) = 2^{2m-1}\sqrt{\pi}\,\delta^{2m}\,\beta(\beta-1)\ \ldots\ [\beta-(m-1)], \quad m \geqslant 1;$$

for n odd we have

$$\sigma_{2m-1} = \lim_{\omega\to 0} I_{2m-1}^{(2)}(\omega,\ \delta) = (-1)^m\,2^m\,\delta^{2m-1}\sum_{k=1}^{m} 2^{k-2}\,\Phi_0^{(k-1)}\left(\frac{1}{s^2}\right)\times$$

$$\times s^{2m-2k+1}(2m-2k-1)!! + (-1)^{m+1}(2\delta)^{2m-1}\int\limits_{0}^{\frac{1}{s}} \Phi_0^{(m)}(y^2)\,dy, \tag{19}$$

where $1!! = (-1)!! = 1$; $\Phi_0(x) = (1-x)^{\beta}$.

On the basis of the relations (11), (12), (15), and (17) the asymptotic expansion for $I_0(\omega,\ \delta)$ assumes the form

$$I_0(\omega,\ \delta) \sim \frac{\sqrt{\pi}}{2} + \sum_{m=1}^{\infty}\frac{\sigma_{2m}}{(2m)!}\,\omega^{2m} + \sum_{m=1}^{\infty}\frac{\sigma_{2m-1}+\chi_{2m-1}}{(2m-1)!}\,\omega^{2m-1}. \tag{20}$$

From the formulas (18) and (19) it is evident that each of χ_{2m-1} and σ_{2m-1} depends on the auxiliary parameter s; however, their sum $(\sigma_{2m-1} + \chi_{2m-1})$ is independent of s, since

$$\frac{d}{ds}\,(\sigma_{2m-1}+\chi_{2m-1}) = 0.$$

We have obtained estimates for the coefficients σ_{2m}, σ_{2m-1}, and χ_{2m-1}, from which there follows the absolute and uniform convergence of the series (20) with respect to ω. We can therefore replace the correspondence sign in the relation (20) by an equality sign, to within a term of the class M_∞.

The asymptotic expansion of the thermal potential $W_0[k\tau^\beta]$ is found with the aid of the relations (13) and (20).

Thus, our method has yielded an asymptotic expansion of the thermal potential $W[\mu(\tau)]$, where $\mu(\tau) = k_1\tau^{\beta_1}$, $\beta_1 > -1$.

NOTATION

u, temperature; a, thermal diffusivity; t, time; x, space coordinate; $\alpha(t)$, boundary motion law; $\varphi(t)$, $\psi(t)$, temperature distributions at the boundary; $W_0[\nu(\tau)]$, $W[\mu(\tau)]$, thermal potentials of a double layer $\nu(t)$, $\mu(t)$, densities of thermal potentials; $H_n(x)$, Hermite polynomials.

LITERATURE CITED

1. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, 2nd ed., Oxford University Press, New York (1959).
2. S. N. Kharin, "Solution of a singular Volterra integral equation with limiting characteristic numbers," in: Conference on Current Progress in Mathematics and Mechanics [in Russian], Alma-Ata (1967).
3. S. N. Kharin, "Asymptotic behavior of a singular integral equation," in: Proceedings of the First Scientific Conference of Young Scientists [in Russian], Izd. Akad. Nauk KazSSR, Alma-Ata (1968).